Fourier's number Fo for different values of Biot's number Bi. In the calculations we took $t_{in} = 20°C$ and $t_0 = 300°C$. The results are presented in the form of plots in Figs. 1 and 2. The solid curves correspond to the solution of the nonlinear boundary-value problem of heat conduction with Kirchhoff's variable, calculated from the formula (9), while the dot-dashed curves were calculated with Kirchhoff's variable taken in the form (10), i.e., they represent the approximate solution. The dashed lines correspond to the solution of the linear boundary-value problem, i.e., under the assumption that the thermophysical characteristics do not depend on the temperature.

The numerical calculations showed that when the temperature dependence of the thermophysical characteristics is taken into account the temperature is lower than the corresponding temperatures in a uniform heat-insensitive body. It should also be noted that linearizing the boundary condition of the third kind, performed by simply replacing t with ϑ, results in a significant distortion of the behavior of the temperature in the heat-sesitive sphere even in the case when the characteristics are linear functions of the temperature with a comparatively small value of the coefficient k_0.

NOTATION

t, temperature field; ϑ, Kirchhoff's variable; λ, thermal conductivity; c_V, heat capacity at constant volume; a, heat-transfer coefficient of the surface $r = R$; $S_{\pm}(\xi)$, asymmetric unit step functions; τ, time; and r, radial coordinate.

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VARIATIONAL FORMULATION OF A NONSTATIONARY HEAT-CONDUCTION PROBLEM

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A functional for a heat conduction problem is determined from thermodynamic considerations; also a class of functions is determined in which it is possible to realize an extremum of the functional.

According to the second law of thermodynamics, in a nonstationary thermodynamical physical system only such spontaneous processes are possible which bring a system of bodies participating in heat exchange closer to equilibrium. In addition, the rate of approach to equilibrium is determined by an increase of entropy s, which reaches a maximum value in the state of equilibrium. In the case of a variational formulation of a nonstationary heat conduction problem, arbitrarily small variations in the solution ϑ can be regarded as the result of the action of some fictitious sources of heat in the system. If the solution ϑ is varied in such a way that the fictitious forces lead to a change of entropy in a direction away from equilibrium with respect to real processes, then, for the same initial and boundary conditions, the time for the system to approach equilibrium will increase. Then the largest rate of approach to equilibrium will correspond to the solution ϑ.

In view of the above, it is appropriate to assume that for the nonstationary heat conduction problem a functional can be constructed having an extremum at the solution ϑ, real-

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izable in some class of approximating functions. As the analysis below indicates, a functional for the problem in question can be obtained through use of the concept of entropy flow s'_τ, defining change of entropy s per unit time, from the condition $s'_\tau = -q/T$. However, as further studies show, the computational method can be simplified if, instead of entropy, we introduce a closely related thermodynamic function Ψ, depending on the heat flux q and temperature ϑ in such a way that the change in Ψ over the time interval $\tau_2 - \tau_1$ amounts to

$$\Delta\Psi = -\int_{\tau_1}^{\tau_2} q\vartheta d\tau. \tag{1}$$

In a thermodynamic sense the quantity Ψ can be regarded as a calculating function, one which makes it possible to determine thermal flow q from the relation $\Psi_\tau' = -q\vartheta$. Like the entropy flow s'_τ, the derivative Ψ_τ' tends towards zero as equilibrium is approached. Therefore, the quantities s'_τ and Ψ_τ' characterize the approach to equilibrium in an analogous way. The difference between s'_τ and Ψ_τ' consists of the fact that in determining Ψ_τ' the temperature ϑ is a factor, whereas in the relationship $s'_\tau = -q/T$ the absolute temperature is found in the denominator.

Upon taking relation (1) into account and also the Fourier heat conduction law $q = -\lambda(\vartheta)\vartheta'_x$, we determine the change in Ψ for a one-dimensional thermal flow and a three-dimensional interval (x_1, x_2) at an element of unit area:

$$\Delta\Psi = \int_{\tau_1}^{\tau_2}\int_{x_1}^{x_2}\left[\lambda(\vartheta)\vartheta_x'^2 + \frac{\partial}{\partial x}(\lambda(\vartheta)\vartheta_x')\vartheta\right] dx d\tau. \tag{2}$$

It follows from relation (2) that Ψ is a thermodynamic function which depends only on the temperature ϑ. Therefore, upon determining $\partial/\partial x(\lambda(\vartheta)\vartheta'_x)$ from the heat conduction equation with source $q_V(\vartheta)$:

$$\frac{\partial}{\partial x}(\lambda(\vartheta)\vartheta_x') - c(\vartheta)\rho\vartheta_\tau' - q_v(\vartheta) = 0, \tag{3}$$

we obtain, taking into account the variation of Ψ on the boundaries,

$$I(\vartheta) = \int_{\tau_1}^{\tau_2}\left\{\int_{x_1}^{x_2}[\lambda(\vartheta)\vartheta_x'^2 + c(\vartheta)\rho\vartheta_\tau'\vartheta + q_v(\vartheta)\vartheta]\,dx + (q\vartheta)\Big|_{x_1}^{x_2}\right\} d\tau = 0. \tag{4}$$

We assume that for the boundary surface S_1 the thermal flow q_1 is given as a function of the temperature ϑ_{S1}, while on the surface $S_2 = S - S_1$ the temperature distribution ϑ_{S2} is known:

$$q_1 = q(\vartheta_{s1}), \quad \vartheta_2 = \vartheta_{s2}. \tag{5}$$

Generalizing $I(\vartheta)$ for a three-dimensional region Ω bounded by surface S, we determine from relations (4) and (5) a functional whose extremal properties we shall investigate:

$$I(\vartheta) = \int_{\tau_1}^{\tau_2}\left\{\int_\Omega [\lambda(\vartheta)\vartheta_i'^2 + c(\vartheta)\rho\vartheta\vartheta_\tau' + q_v(\vartheta)\vartheta]d\Omega - \int_{S_1} q(\vartheta_{s1})\vartheta_{s1}dS_1 - \right.$$

$$\left. - \int_{S_2}\lambda(\vartheta_{s2})(\vartheta_{s2})_n'\vartheta_{s2}dS_2\right\} d\tau = 0; \quad \vartheta_i' = d\vartheta/dx_i \ (i = 1, 2, 3). \tag{6}$$

From relations (5) and (6) we obtain with the aid of the Gauss–Ostrogradskii formula

$$I(\vartheta) = \int_{\tau_1}^{\tau_2}\int_\Omega L(\vartheta)\vartheta d\Omega d\tau = 0, \tag{7}$$

where

$$L(\vartheta) = c(\vartheta)\rho\vartheta_\tau' - \frac{\partial}{\partial x_i}(\lambda(\vartheta)\vartheta_i') - q_v(\vartheta) = 0. \tag{8}$$

Presence of the operator $L(\vartheta)$ in relation (7) from the differential equation of heat conduction is the result of a generalization of Eq. (3) into Eq. (4) on the region Ω.

Functional (6) represents a variational formulation of the nonstationary heat conduction problem. Here the boundary conditions are taken into account by the last two terms in Eq. (6).

If, instead of a variation of the function $\Psi_\tau' = -q\vartheta$ in relation (1), we consider a change in the entropy of the thermal flow, $s_\tau' = -q/T$, the integrand expression in relation (7) then assumes the form $L(\vartheta)/T^2$. For such a functional, the coefficients used in defining the approximating functions will depend on the values of the absolute temperature T_m of the medium and the initial temperature T_0, thereby substantially complicating the search for these coefficients. In what follows we shall use functional (6) and determine relative values of the temperature $\vartheta = T - T_0$, during heating, or $\vartheta = T - T_m$, during cooling, thereby enabling us to eliminate the influence of the origin of reference of the temperature on the value of the coefficients.

We shall vary the solution ϑ of the problem in some class of functions Θ to which there correspond arbitrarily small variations of f:

$$\Theta=\vartheta+f,\ \Theta,\ f\in C^2. \tag{9}$$

In accordance with relations (7) and (8), upon substituting solutions ϑ of the problem into functional (6), we find that the functional vanishes:

$$I(\Theta)=0. \tag{10}$$

Therefore, we verify the presence of an extremum of functional (6) by determining the sign of its increment ΔI, which, according to Eq. (10), is equal to the value of functional (6) under the variations (9): $\Delta I(\Theta) \equiv I(\Theta)$.

The approximating functions Θ cannot satisfy equation (8), to which there corresponds the residual $\varepsilon(x_i, \tau)$:

$$L(\Theta)-\varepsilon(x_i, \tau)=0. \tag{11}$$

If functions Θ do not satisfy boundary conditions (5), we then have the residuals

$$E_1= q(\Theta_{s1}) - \lambda(\Theta_{s1})(\Theta_{s1})_n';\ E_2=\vartheta_{s2} - \Theta_{s2}. \tag{12}$$

The quantity ε in Eq. (11) can be regarded as a fictitious heat source in region Ω. The residual E_1 also represents a fictitious heat source on surface S_1, and the quantity E_2 is equal to the change in temperature of surface S_2 as the result of the action of a fictitious heat source on this surface.

We select approximating functions (9) satisfying the inequality

$$I(\Theta)<0. \tag{13}$$

Then, according to relations (9) and (10), to condition (13) there will correspond a maximum of functional (6). In order to verify that a maximum in relation (13), realizable in the class of functions Θ, exists on a solution ϑ, it is sufficient to convince ourselves that it is unique and that values of the functional in neighborhoods of the extremal point will be arbitrarily small. We assume here that uniqueness of a solution ϑ is determined by physical conditions. According to relations (6) and (10)-(13), for approximations Θ the change of function Ψ in region Ω will be less than on the boundaries. Maximum values of the change in Ψ, and of its derivative with respect to the time, $\Psi_\tau' = -q\vartheta$, in Ω in relation to their values on the boundaries occur for a solution ϑ, which determines the existence of an extremum in relations (10), (13).

Analysis shows that it is not possible to give a physical meaning to functional (6) using elementary concepts. In some problems an extremum of the functional can exist to which a physical interpretation can be given according to the meaning of the problem, for example, in the determination of a minimal surface or maximum entropy. We shall consider variations for which the extremum (10), (13) is unique and values of the time corresponding to the solution ϑ are less than for the approximations Θ. Then the rate of change of the temperature ϑ, being defined in accordance with relation (6) for given heat exchange conditions on the boundaries (5), will have the largest value. If such an extremum can be realized in some class of functions, it then becomes possible to assume that under the conditions considered some maximally possible rate of approximation to equilibrium can be the result of physical peculiarities of real processes. In particular, this agrees with the second law of thermodynamics, meaning that the existence of only such spontaneous processes is possible which bring a system to equilibrium.

If we take into account the statistical nature of heat transfer in macrosystems, it might be possible to confirm the assumption made if we can show that the distribution of temperature ϑ corresponding to the solution of the problem is statistically most probable with respect to arbitrary approximations Θ. However, with the aforesaid taken into account, there are reasons for assuming, as in the case of the second law of thermodynamics, that the existence of an extremum, to which there corresponds physically a minimum time, cannot be established from arbitrary theoretical considerations or formal proofs and must be based on the results of observations with physical objects. For the problem in question, this can be done, through computational verification, whether there exist classes of functions in which conditions (9) and (13) are uniquely satisfied for all values of x_i and τ. In proceeding, we shall obtain approximations to the solution ϑ from above and below, enabling us to verify the existence of an extremum of functional (6) and its uniqueness in each concrete problem.

We select functions Θ such that at point Θ_ν of a maximum in relation (13) the absolute value of functional (6) is the smallest possible and sufficiently small:

$$|I(\Theta_\nu)| < \xi_1. \tag{14}$$

We assume that this maximum is unique in the class of functions Θ, an assumption to be corroborated in subsequent calculations. Taking into account uniqueness of a solution ϑ, we can assume existence of a class of functions in which the extremum in relations (10), (13) on solutions ϑ is also attained in a unique way. Then, taking relations (10) and (13), into account, we assume that the function Θ_ν, satisfying condition (14), will be a better approximation to solution ϑ. In addition, values of the moduli $|I(\Theta_\nu)|$ must be sufficiently small for all values of τ^*, $\tau_1 < \tau^* \leq \tau_2$.

In accordance with the aforesaid, one must assume that values of the approximating functions Θ_i, satisfying inequalities (13) and (14), as well as the condition $|\Theta_i - \Theta_\nu| < \xi_2$, can be considered to be the result of arbitrarily small variations of f on the solution ϑ. With the aim of verifying this supposition, we shall determine the signs of the variations of f corresponding to the approximating functions Θ_i; this also enables us to obtain approximations Θ_a and Θ_b, from below and above, respectively, to solution ϑ.

We obtain a fictitious thermal flow on surface $S = S_1 + S_2$, due to the action of heat sources ε, E_1, and E_2:

$$\Delta q_s = \int_{\tau_1}^{\tau_2} \left[\int_{\Omega} \varepsilon d\Omega - \int_{S_1} E_1 dS_1 - \int_{S_2} (q(\vartheta_{s2}) - q(\Theta_{s2})) dS_2 \right] d\tau. \tag{15}$$

According to the law of conservation of energy, variations of the surface temperature are determined by values of thermal flows Δq_s at corresponding points of the boundary surface. If, in addition, signs of the thermal flows Δq_s in Eq. (15) do not vary over the time interval $\tau_2 - \tau_1$, then the signs of variations $f_a = \Theta_a - \vartheta$ and $f_b = \Theta_b - \vartheta$, relation (9) will depend only on signs of the thermal flows $\Delta q_s(\Theta_a)$ and $\Delta q_s(\Theta_b)$. For an approximation Θ_a from below to solution ϑ the flow $\Delta q_s(\Theta_a)$ will have a negative value, while for an approximation Θ_b from above to ϑ the flow $\Delta q_s(\Theta_b)$ will be positive:

$$\Delta q_s(\Theta_a) < 0, \ \Delta q_s(\Theta_b) > 0, \ \tau_1 < \tau^* \leqslant \tau_2. \tag{16}$$

Then, upon selecting approximating functions Θ_i, with inequalities (16) taken into account, we obtain the relations

$$f_a < 0, \ f_b > 0, \ \Theta_a < \vartheta < \Theta_b, \tag{17}$$

which signify that the values of Θ_a and Θ_b are approximations to the solution ϑ from below and above, respectively.

Upon substituting into Eq. (15) solutions ϑ of problem (5) and (8), we have the equation $\Delta q_s(\vartheta) = 0$ for all values of the time τ. Therefore, we select functions Θ_ν, taking into account the condition

$$\Delta q_s(\Theta_\nu) = 0, \tag{18}$$

to which there will correspond a best approximation Θ_ν to solution ϑ.

Functions Θ_ν cannot satisfy Eq. (18) for all values of τ^*, $\tau_1 < \tau^* \leq \tau_2$. In such a case we obtain approximations Θ_ν through obtaining minimum values of the moduli $|\Delta q_s(\Theta_\nu)|$ for different instants of time τ^*:

$$|\Delta q_s(\Theta_\nu)| < \xi_3, \ \tau_1 < \tau^* \leqslant \tau_2.$$

Here it is necessary to verify that for all values of τ^* the inequalities (16) are satisfied. The directions of the fictitious thermal sources $\Delta q_s(\Theta_a)$ and $\Delta q_s(\Theta_b)$ are then maintained for all values of τ^* and the inequalities (17), in accordance with the law of conservation of energy, will be satisfied. Values of the moduli $|I(\Theta_\nu)|$ must also be determined in a similar way.

For an accepted error δ_1 in calculating the temperature, we obtain

$$\delta = (\Theta_b - \Theta_a)\frac{1}{\Theta_a} < \delta_1. \tag{19}$$

The difference $\Theta_b - \Theta_a$ in relation (19) and the error corresponding to it depend on the choice of approximating functions Θ, which must satisfy conditions (16). If the functions Θ do not guarantee a sufficiently good approximation to the solution ϑ, then, as calculations show, the inequalities (16) can only be satisfied for large values of the difference $\Theta_b - \Theta_a$.

An estimate of the signs of the variations proved to be possible because the values of the sources ε, E_1, and E_2, in contrast to the variations f, are not arbitrary and can be determined depending on the choice of functions Θ_a and Θ_b, which constitute solutions of problem (11) and (12) having a physical meaning. This makes it possible to find the fictitious flows caused by the sources ε, E_1 and E_2, and also to establish the signs of these flows, which determine the signs of the variations f_a and f_b. If the system of approximating functions Θ is not complete, then an estimate of the error in the calculation through a formal analysis of variations without use of the law of conservation of energy cannot, evidently, be carried out.

As approximations Θ we select functions (9) with constant coefficients. Then, after integration, functional (6) becomes a function of these coefficients $I(\beta_1, \beta_2, \ldots, \beta_j, \tau_m)$. If the integration is carried out only over the region Ω, then for fixed m values of τ_2 we obtain the analogous function $I(\beta_1, \beta_2, \ldots, \beta_j, \tau_m)$.

The unknown coefficients β can be determined from the conditions for a minimum sum of moduli

$$\sum_{i=2}^{h} |I(\tau_i)| = I_{\min}; \quad \sum_{h+1}^{j} |\Delta q(\tau_i)| = \Delta q_{s\min}. \tag{20}$$

These relations also make it possible to obtain values of β for the case in which some of the equations do not have roots close to the extremal point.

According to relations (6) and (10)-(12) existence of a minimum for functional (6) is determined by the inequality

$$I(\Theta) > 0. \tag{21}$$

We select approximating functions Θ in such a way that inequalities (14), (17), and (18) preserve their sense. Following the reasoning presented above, we find that approximations Θ_ν can be determined in a similar manner for inequality (21).

To calculate the temperature at some point r inside region Ω, we break up this region into two subregions Ω_1 and Ω_2 in such a way that point r is located on the common boundary S_r between Ω_1 and Ω_2. We assume that on boundary S_r there exists a fictitious heat source $E_r = q_{r2} - q_{r1}$. If the quantities ε and E_r are of opposite sign, then, as a result of the action of the boundary source E_r, the influence of the internal heat source ε in Ω on the temperature of the partitioning surface S_r will diminish.

Surface S_r differs from S by the presence of a single fictitious source, whereas there are two sources acting on S. Therefore, all the discussion for approximations Θ_v in region Ω holds true for subregion Ω_1. If coefficients β_j are varied over the region, the approximation will then be realized in the form of a polygonal line, as in the case of approximation by splines.

Functional (9) contains no initial conditions, which means that in a variational formulation of a nonstationary heat conduction problem the initial conditions play no role. As initial temperature here, we can consider the distribution of temperature Θ_1 in region Ω at an arbitrary instant of time $\tau_0 \equiv \tau_1$. In the sequel we shall take into account the initial conditions, selecting functions Θ in a class satisfying these conditions.

As an example, we consider the problem of radiative heating of an unbounded plate of unit thickness with thermally insulated surface $\Theta_x'(0, \tau) = 0$ at a constant absolute temperature T_m = const of the surrounding medium and a uniformly distributed initial plate temperature: $T(x, 0)$ = const.

We use a trigonometric polynomial to approximate the absolute temperature $T(x, \tau)$ of the plate:

$$\Theta(x, \mathrm{Fo}) = 1 - \gamma\left[\sum_{i=1}^{n} D_i(\mathrm{Fo})\cos\mu_i x - N(\mathrm{Fo})\cos kx\right]; \tag{22}$$

$$\Theta(x, \mathrm{Fo}) = T(x, \mathrm{Fo})/T_m;\ \gamma = 1 - \Theta(x, 0);\ \mathrm{Fo} = a\tau; \tag{23}$$

$$D_i(\mathrm{Fo}) = D_i\exp[-\varphi_i\mathrm{Fo} + b\exp(-40\mathrm{Fo})];\ N(\mathrm{Fo}) = B\exp(-\varphi_0\mathrm{Fo}). \tag{24}$$

We obtain the amount of heat flow at the boundary x = 1 from the Stefan—Boltzman law:

$$q(1, \mathrm{Fo}) = CT_m^4[1 - \Theta^4(1, \mathrm{Fo})]. \tag{25}$$

Using relations (6), (22), (25), and the boundary condition $\Theta_x'(0, \mathrm{Fo}) = 0$, we determine the functional

$$\frac{1}{\lambda T_m^2} I(\Theta) = \int_0^1 \left[\Theta_x'^2(x, \mathrm{Fo}) + \frac{1}{a}\Theta(x, \mathrm{Fo})Q_\tau'(x, \mathrm{Fo})\right]dx -$$

$$- \mathrm{Bi}_p\Theta(1, \mathrm{Fo})[1 - \Theta^4(1, \mathrm{Fo})];\ \mathrm{Bi}_p = CT_m^3/\lambda. \tag{26}$$

We select the terms $D_i(0)\cos\mu_i x$ in the class of orthogonal functions and obtain D_i from the initial condition $\Theta(x, 0) = 1 - \gamma$:

$$D_i = \frac{(2\sin\mu_i + BP_i\mu_i)\exp b}{\mu_i + \sin\mu_i\cos\mu_i};\ P_i = \frac{\sin(\mu_i + k)}{\mu_i + k} + \frac{\sin(\mu_i - k)}{\mu_i - k}. \tag{27}$$

For choice of initial approximation we specify the Biot number from boundary condition (25):

$$\mathrm{Bi} = \frac{\alpha}{\lambda} = \frac{q[\Theta(1, \mathrm{Fo})]}{\lambda T_m[1 - \Theta(1, \mathrm{Fo})]} = \mathrm{Bi}_p[1 + \Theta(1, \mathrm{Fo})][1 + \Theta^2(1, \mathrm{Fo})].$$

When $\tau = \infty$, $\Theta(1, \infty) = 1$ and $\mathrm{Bi} = 4\mathrm{Bi}_p$; therefore, we obtain the initial values μ_{10} and $\varphi_{10} = \mu_{10}^2$ from the characteristic equation for the linear problem when Bi = const: $\mu_t\tan\mu_i = 4\mathrm{Bi}_p$ [1].

We select coefficients φ_i for i > 2, $\varphi_i = \varphi_2(2i + 1)^2/(2i - 1)^2$. Then for i > 2 the φ_i will increase rapidly, and when i = 2 the function Θ will approximate the solution ϑ for $\mathrm{Fo} > \mathrm{Fo}_1$.

For determination of the coefficients we obtain from relations (8) and (18), after intergration and some simplifications,

$$\frac{1}{T_m c\rho}\Delta q_s(\Theta) = \int_0^1 \Theta(x, \mathrm{Fo}_1)dx - \frac{1}{T_m c\rho}\int_0^{\tau_1} q_1(1, \mathrm{Fo})d\tau - \Theta(x, 0); \mathrm{Fo}_1 = a\tau_1. \tag{28}$$

TABLE 1. Values of Relative Temperature of Surface of Unbounded Plate Subjected to Radiative Heating

$\mathrm{Bi_p} = 0{,}5;\ \gamma = 0{,}85$						
$\mu_1 = 1{,}1757;\ \varphi_1 = 1{,}0128;\ \mu_2 = 3{,}3516;\ \varphi_2 = 3{,}7310;$ $k = 1{,}9346;\ b = 0{,}0030;\ B = 1{,}0754;\ \varphi_0 = 1{,}8929$						
Fo	0,3	0,5	0,7	1,0	1,5	2,0
Θ_s	0,4571	0,5410	0,6228	0,7257	0,8414	0,9079
Θ_{s1}	0,4532	0,5446	0,6238	0,7231	0,8396	0,9090
$\mathrm{Bi_p} = 2{,}0;\ \gamma = 0{,}8$						
$\mu_1 = 1{,}3975;\ \varphi_1 = 1{,}9540;\ \mu_2 = 4{,}2303;\ \varphi_2 = 17{,}86;$ $k = 2{,}0431;\ b = 0{,}0335;\ B = 0{,}1295;\ \varphi_0 = 4{,}1698$						
Fo	0,3	0,4	0,5	0,6	0,8	1,0
Θ_s	0,8784	0,9026	0,9214	0,9363	0,9578	0,9719
Θ_{s1}	0,8774	0,9026	0,9215	0,9364	0,9579	0,9719
δ, %		0,1				0,1

Upon taking into account the quantity of heat $T_m c\rho\gamma$, accumulated per unit volume of the plate at $\tau = \infty$, and the condition (25), we obtain the following for the last integral in Eq. (28) when $\mathrm{Fo} > \mathrm{Fo}_1$ and $i = 2$:

$$\frac{1}{T_m c\rho}\int_0^{\tau_1} q(1,\ \mathrm{Fo})\,d\tau = \gamma - \mathrm{Bi_p}\int_{\mathrm{Fo}_1}^{\infty}[1 - \Theta^4(1,\ \mathrm{Fo})]\,d\mathrm{Fo};\ c\rho\gamma T_m = \int_0^{\infty} q(1,\ \mathrm{Fo})\,d\tau.$$

We obtain the unknown coefficients in relations (22)-(24) beforehand from the system of equations $I(\tau_j) = 0$ and $\Delta q_s(\tau_j) = 0$ when Fo varies from 0.4 to 1.0; this takes 3 to 4 iterations. We then revise all the coefficients from conditions (18) and (20) using Rozenbrok's multidimensional optimization method [2]. In the course of our calculations we vary μ_1; from $\Delta q_s(\tau_j) = 0$ we verify the value of φ_1 to which I_{min} must correspond.

Table 1 gives values for the temperature Θ_s of the surface of the plate for $\mathrm{Bi_p} = 0.5$ and $\mathrm{Bi_p} = 2.0$, obtained from relations (20), (26), and (28); it also gives values for the error δ. A comparison of the results of the variational calculations with values of the temperature Θ_{s1}, obtained by the method of finite differences [3], shows that when $\mathrm{Fo} > 0.3$ both methods have the same order of error. Sums of moduli, obtained from relations (20) for three values of Fo (0.5, 0.75, and 1.0) and $\mathrm{Bi_p} = 2.0$, were found to be $I_{min} = 0.0011$ and $\Delta q_{smin} = 0.0002$; these values confirm that when $\mathrm{Bi_p} = 2.0$ expression (22) gives a good approximation to the solution of the problem.

The variational method we have presented makes it possible not only to represent results of our calculations in analytic form, but it also allows us to carry out integration with respect to certain variables in the process of treating the method numerically. The latter can prove to be important for multidimensional nonlinear problems, particularly, for the solution of the heat conduction equation, along with other differential equations describing some process. Determination of the coefficients in relations (22)-(24) by the variational method required an operative memory on the order of 2.5 Kbytes on the microcomputer HP-41CV. Here, instead of solving a system of nonlinear equations, we carried out an optimization of conditions (20); this substantially broadens possibilities of the method.

In using direct methods for functional (6) errors may appear which are common for variational problems and are connected with possible violations of certain correctness conditions owing to the arbitrariness of the variations (9). However, in the case of functional (6), an estimate of the errors arising from there can be ascertained from thermodynamic conditions.

As the following calculations show, some coefficients, useful in determining the approximating functions Θ, in particular, coefficients depending on the time, can, close to an extremum point, take on minimum or maximum values, a situation which makes their retrieval easier. If in relation (6) we use functions $\Theta \equiv \vartheta$, which satisfy relations (5) and (8), it is easy to see that coefficients obtained from relations (17) and (20) generally coincide with values of the coefficients obtained from characteristic equations by classical methods. For nonlinear problems there exist residuals in relations (11) and (12); however, the functions Θ can always be improved. If it is not possible to select functions of another kind, we can then increase the number of defining coefficients and points corresponding to them at which an approximation of functions Θ to the solution can be effected.

Necessary and sufficient conditions for the existence of an extremum of functional (6) are determined by relations (10), (13), (17), and (21), and, therefore, no proof of corresponding lemmas from variational calculus is needed. Realization of an extremum of functional (26) in the class of functions (22)-(24) can be considered as a confirmation of the existence of conditions (14)-(18) on the solution of boundary value problem (5) and (8). If an extremum of this physical type exists, then, as analysis of classical variational problems shows, the choice of approximating functions and the determination of unknown coefficients are substantially simplified. The approximation Θ can be effected with the aid of various functions, but calculations show that a satisfactorily chosen class of functions turns out to be unique in the sense of relations (14)-(18). Existence of such functions for problem (5) and (8) can be regarded as a consequence of the uniqueness of its solution, arising, in turn, from physical conditions. According to relations (2) and (4), for a heat conduction process in a solid body, Ψ is a function of state and, therefore, its rate of change Ψ_τ', as well as the change of entropy of flow $s_\tau' = -q/T$, uniquely determines the heat transfer process. With the aforesaid taken into account, there is a basis for assuming that the extremum (13) or (21), to which the specified quantity Ψ_τ' corresponds, is also unique.

Functional (6) makes it possible to carry out calculations, using as initial temperature its value $\Theta(x_i, \infty)$ at $\tau = \infty$. Such a condition assumes passage of heat from a lower to a higher temperature and therefore contradicts the second law of thermodynamics. Since the relations obtained above, in this connection, do not change, a reading of the temperature from $\Theta(x_i, \infty)$ can be used as a computational device. If in sequence (22) the values of the functions decrease rapidly, then, in integrating between the limits from ∞ to Fo_1, we cannot take into account terms which become sufficiently small for large Fo_1, whereas these terms do influence the result of integrating between the limits of 0 and Fo_1. A similar calculation was carried out in computing the last integral in Eq. (28).

A comparison of integrals used in the solution of boundary value problems by the Galerkin method with integral (7) shows that their integrands are similar. Let us substitute the approximation solution Θ_ν into relation (7) and assume that $I(\Theta_\nu) = 0$. In this case equation $I(\Theta_\nu) = 0$ determines orthogonality of residual ε in relation (11) and the function Θ_ν. In the Galerkin method integrals of the form (7) are obtained from conditions of orthogonality of residual ε and each of the functions in the selected sequence $\Theta_\nu = \Theta_1 + \Theta_2 + \ldots + \Theta_j$. Summation of the integrals obtained in this way leads to the equation $I(\Theta_\nu) = 0$. Then, in accordance with the above, the Galerkin method can be regarded as a variational method if the approximating functions are selected with the same conditions taken into account as in realizing an extremum of functional (6). In solving problem (5) and (8) by the Galerkin method it is very rare that one can satisfy condition (7) for the approximate solution Θ_ν and each function Θ_j, a fact which significantly limits the choice of the approximating functions and does not permit broad use of this method. Condition (14) for functional (6) makes it possible to broaden essentially the class of approximating functions. In addition, functional (6) takes into account the influence of residuals E_1 and E_2 being considered as fictitious heat sources on the boundaries.

NOTATION

ϑ, temperature; Ψ, new thermodynamic function; τ, time; q, thermal flow; x, x_i, coordinates; λ, coefficient of thermal conductivity; q_V, volume heat source; ρ, density; c, thermal capacity; S, boundary surface; Ω, three-dimensional region; ϑ_n', temperature gradient directed along normal to boundary surface; T, absolute temperature; T_m, temperature of medium; I, functional; Θ, approximating function; f, arbitrarily small variations of solution of problem; ε, E_1, E_2, fictitious sources of heat; a, thermal diffusivity coefficient; C, radiation coefficient; Fo = $a\tau$, Fourier number (dimensionless time) for plate of unit thickness; $(1-\gamma)$, relative initial temperature; β_i, μ_i, φ_i, D_i, B, k, b, coefficients defined in text; α, heat transfer coefficient; Bi, Biot number; Bi_p, radiative Biot number for plate of unit thickness.

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EXISTENCE OF SOLITARY WAVES IN A PRESTRESSED NONLINEAR THERMOELASTIC MEDIUM WITH DRY FRICTION

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The one-dimensional problem of the solitary wave propagation in a prestressed nonlinear thermoelastic medium with dry friction is analyzed on the basis of a geometrically nonlinear model. An equation is derived for calculating the free energy at which solitary waves can be generated in such a medium. It is shown that the wave velocity depends on the initial state of the medium and on the dry friction law.

1. BASIC EQUATIONS OF THE NONLINEAR THEORY OF THERMOELASTICITY IN THE PRESENCE OF DRY FRICTION FORCES

Let a body obey the laws of the nonlinear theory of thermoelasticity in the presence of dry friction; the analysis of the wave processes in the one-dimensional problem in Lagrangian variables is then reduced to the solution of the following equations [1-4]: a) the equation of motion

$$\frac{\partial}{\partial x}[(1+\varepsilon)\sigma^*]=\rho_0\frac{\partial^2 u}{\partial t^2}+\operatorname{sgn} v f(|v|)$$

or

$$\frac{\partial^2}{\partial x^2}[(1+\varepsilon)\sigma^*]=\rho_0\frac{\partial^2\varepsilon}{\partial t^2}+\frac{\partial}{\partial x}[\operatorname{sgn} v f(|v|)], \qquad (1)$$

where $\varepsilon = \partial u/\partial x$, $v = \partial u/\partial t$, f is a continuously differentiable function on the interval $(0; \alpha]$ $(\alpha > 0)$, $f'(v)$, $f(v) > 0$ for $v \in (0; \alpha]$, and $f(0) = 0$; the condition $f(0) = 0$ ensures the differentiability of the function sign v $f(|v|)$ $(v \in [-\alpha; \alpha])$; b) the heat-conduction equation, assuming that

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